Combinatorial Game Theory

Analysis of the Game of Nim and some interesting Variants

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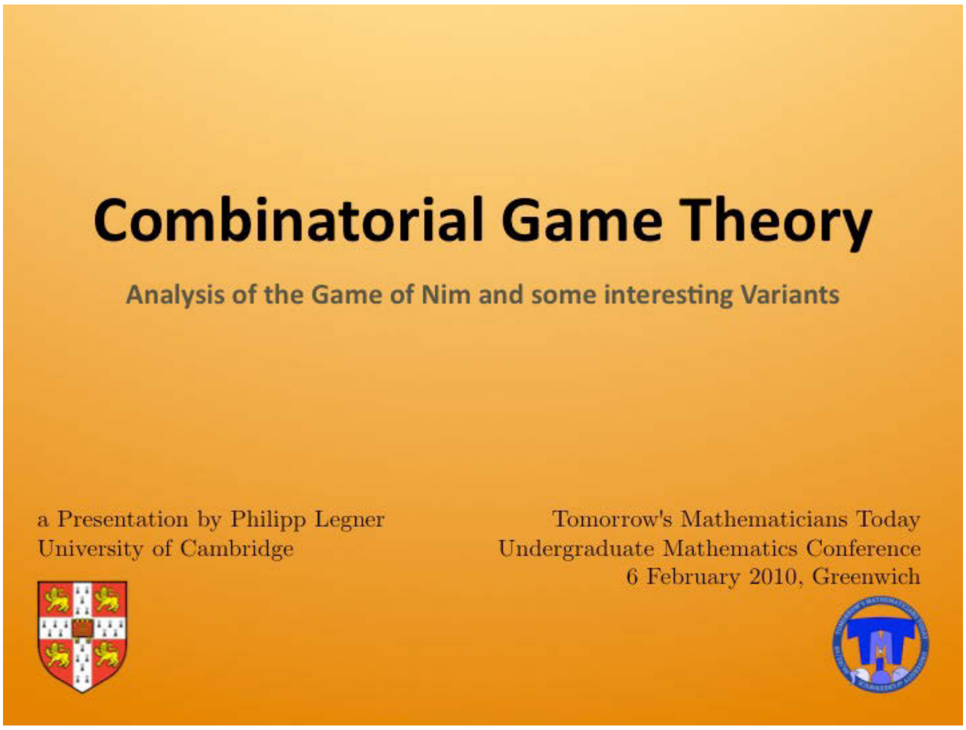
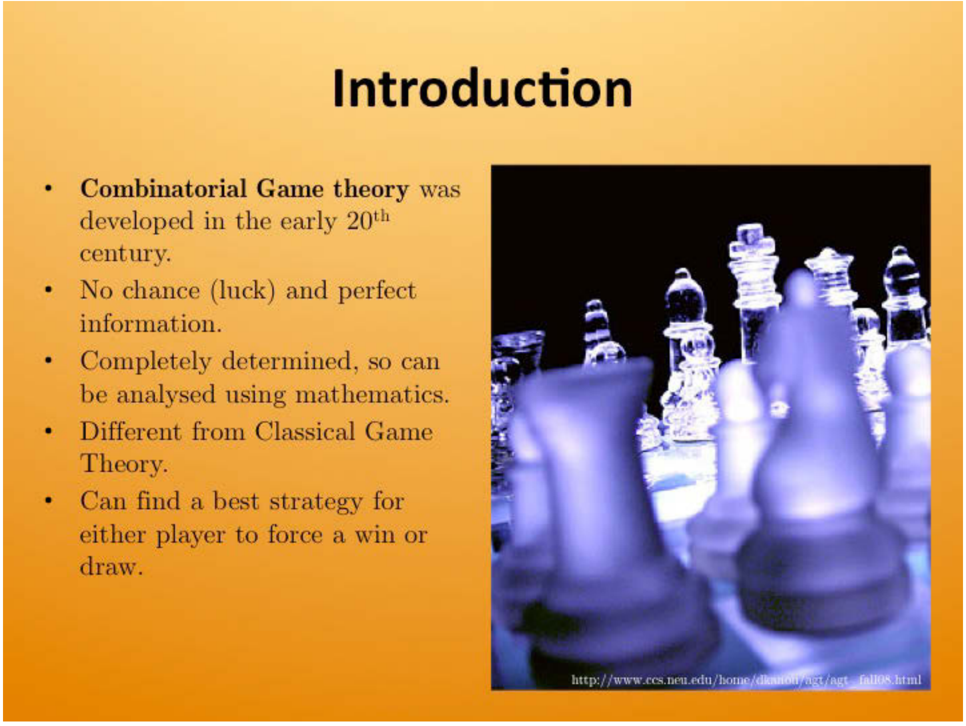
Abstract

Combinatorial game theory is very different from classical game theory, since it doesn't involve chance, cooperation or conflict. Therefore combinatorial games can be analysed completely using mathematical theory. The most famous combinatorial game is Nim: the opponents alternately remove some counters from distinct heaps and the player to remove the last counter wins. During this presentation I want to show how to derive and prove an optimal strategy for either player, which involves graph theory, the Sprague-Grundy function and the binary digital sum of the heap sizes (called the Nim-sum).

I will then prove the Sprague-Grundy theorem, that every impartial game is equivalent to a certain game of Nim. This is fundamental to combinatorial game theory and can be used to analyse many variants of Nim, such as Grundy's Nim.

In the second part of the talk, I want to present "Lucky Nim" – a combination of an impartial game that can be analysed mathematically, and coin tossing. This non-combinatorial game seems to depend on pure chance. However analysing the game using the Sprague-Gundy function gives a very surprising result.



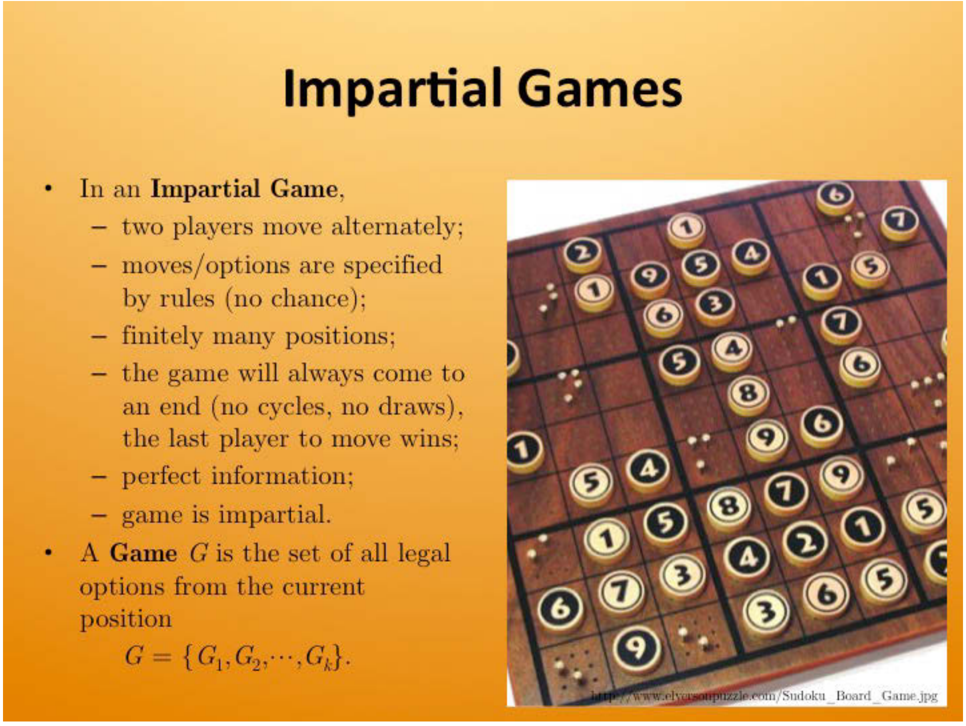


Introduction

Combinatorial Game Theory was formally developed in the early 20$^{\text{th}}$ century, but was of interest to many mathematicians since the ancient Greeks. The key characteristic of combinatorial games is that there is no chance or luck involved and all players have perfect information. This means that the game is completely determined and hence and be analysed using mathematics. Therefore combinatorial game theory is very different from “classical” Game theory, which is concerned with conflicts and cooperation in games and takes a rather probabilistic approach.

It is intuitively clear that there must be a “best” strategy for either player to pursuit and if both players play correctly, the outcome is determined by the initial conditions. In fact, it can be proved that in combinatorial games, either *one* player can force a win, or *both* players can force a draw. The aim of combinatorial game theory is to analyse the games and try to find winning strategies. This gives rise to a very rich and interesting mathematical theory.

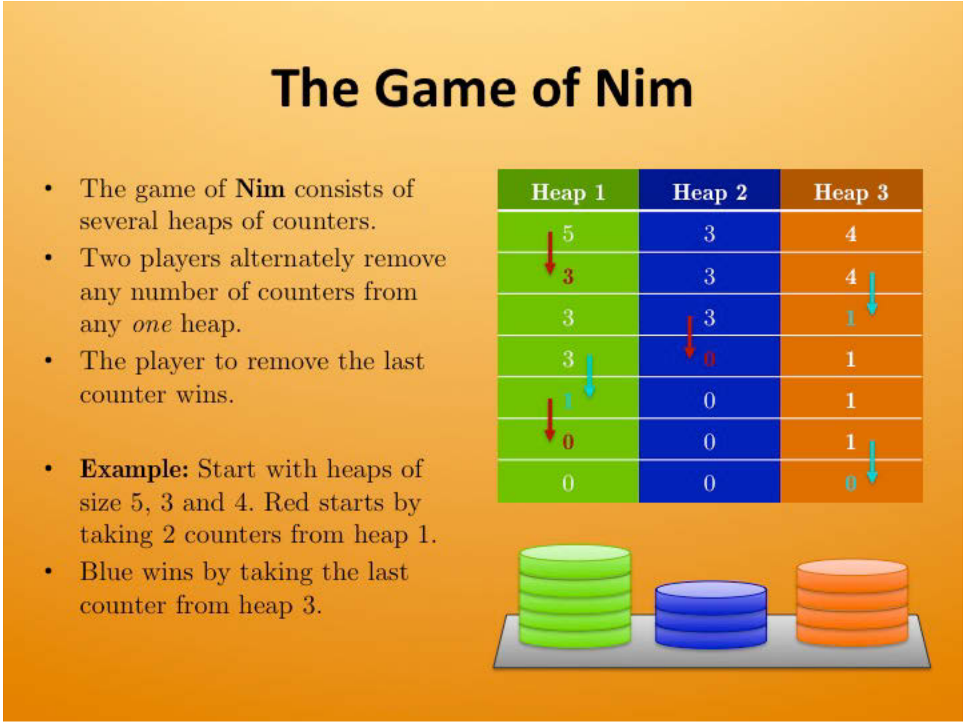
We see that chess, for example, is a combinatorial game. However, chess is so complicated, that it is impossible for computers to evaluate and prove a *best* strategy.



Impartial Games

In fact, I only want to talk about Impartial Games. These satisfy several additional properties: Exactly two opponents move alternately. The moves and all options are clearly specified by rules and there are no chance moves. There are only finitely many different positions and the game will always come to an end when one player is unable to move. This means that there is no draw and no cycles, which could repeat forever. Usually, the last player to move wins (however in the miserè variant of some games, the last player to move loses). There is perfect information and the game is impartial, i.e. from any one position of the game, both players have the same choice of move. We see that chess, for example, is not an impartial game, since – from any position – one player can only move white figures and one player can only move black ones.

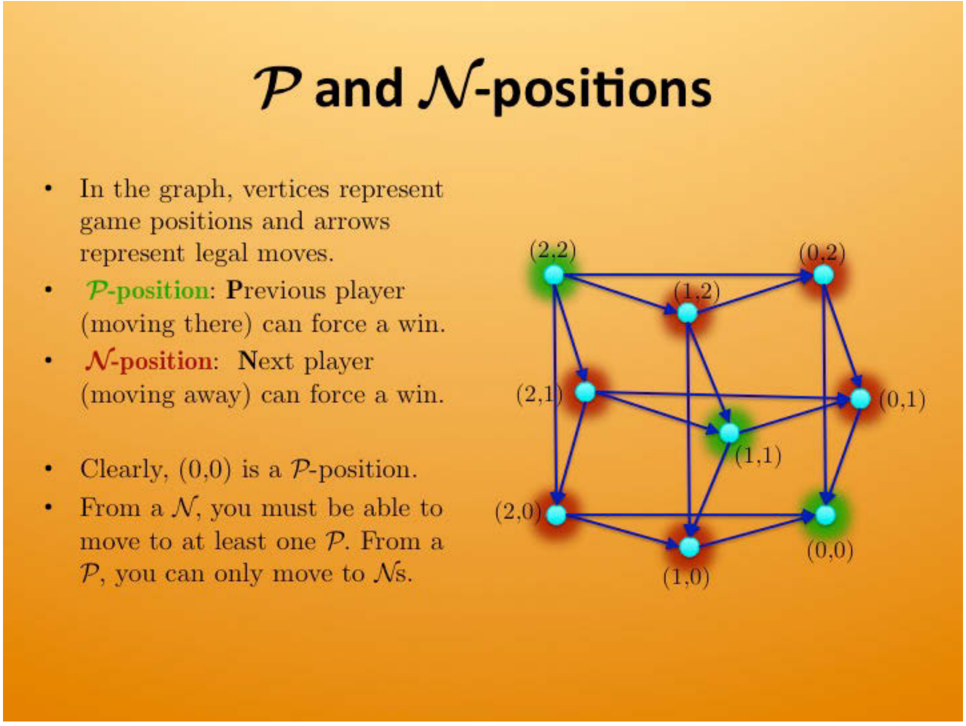
Formally, we can define a Game G recursively as a the set of all legal options $\{G_1, G_2, \ldots, G_k\}$ from the current position. Clearly, the empty set is always the terminal position and whoever reaches this position first wins. Later in this presentation, we will meet another definition of an impartial game using graph theory.



The Game of Nim

The most important and most famous impartial game is Nim. It consists of several heaps of counters and the opponents alternately remove counters from the heaps. Each player can remove as many counters as he wants, but only from *one* heap at a time. He has to remove *some* counters. The player to remove the last counter in total wins.

This is an example of a possible game of Nim: We start with heaps of sizes 5, 3 and 4. Red starts by removing two counters from heap 1. Then blue removes three counters from heap 3. Then red removes all three counters from heap 2. Then blue removes two counters from heap 1 and red takes the last counter from heap 1. Finally, blue takes the last counter from heap 3 and – since there are no more counters left – wins.

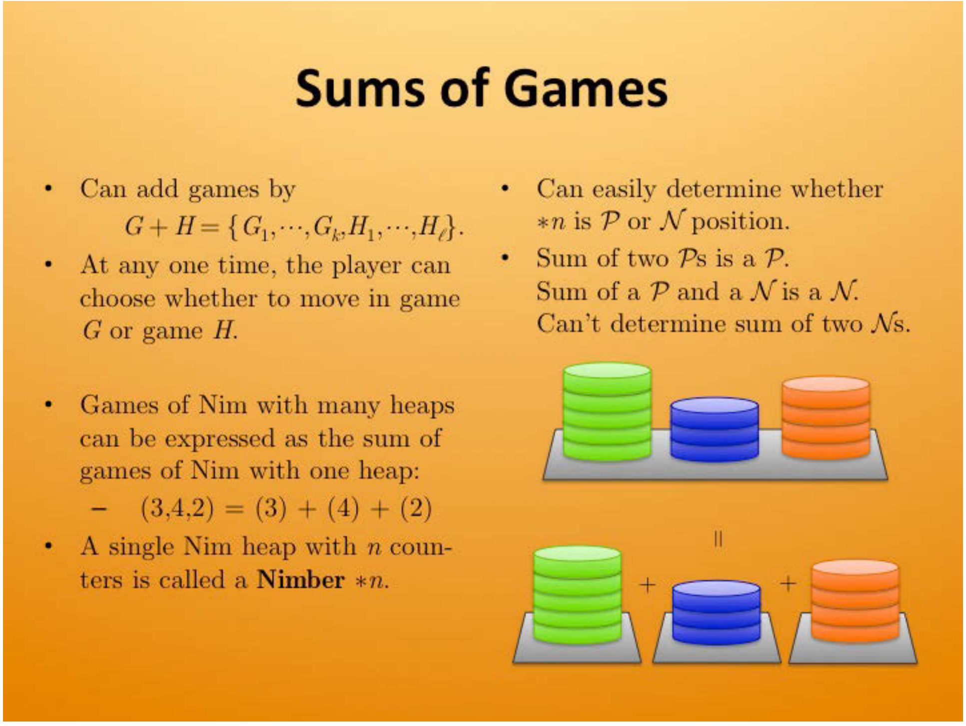


$\mathcal{P}$ and $\mathcal{N}$-positions

To find a winning strategy for Nim, we write down all possible positions of the game as vertices of a graph. The arrows represent legal moves. Since Nim is an impartial game, there must be a winning strategy. However it is clear that the strategy must depend on where on the graph you start and who is the next one to move. We say that a vertex is a $\mathcal{P}$-position, if the previous player, who moves to this position, can force a win; and we say that a vertex is a $\mathcal{N}$-position, if the next player, who moves away from there, can force a win.

Clearly, (0,0) is a $\mathcal{P}$-position, since whoever arrives there has already won. And now we observe, that from any $\mathcal{N}$-position, you must be able to move to at least one $\mathcal{P}$-position (otherwise there is no winning strategy). From any $\mathcal{P}$-position, you can only move to $\mathcal{N}$-positions (otherwise your opponent would also have a winning strategy, which is a contradiction). Now it is easy to see how the outcome is predetermined depending on where you start.

Note that for a game of Nim with only two piles, all $\mathcal{P}$-positions are two piles of the same size: This is clear, because from there on, the winning strategy is to repeat your opponents moves on the other pile respectively. This way, you will always take the last counter.

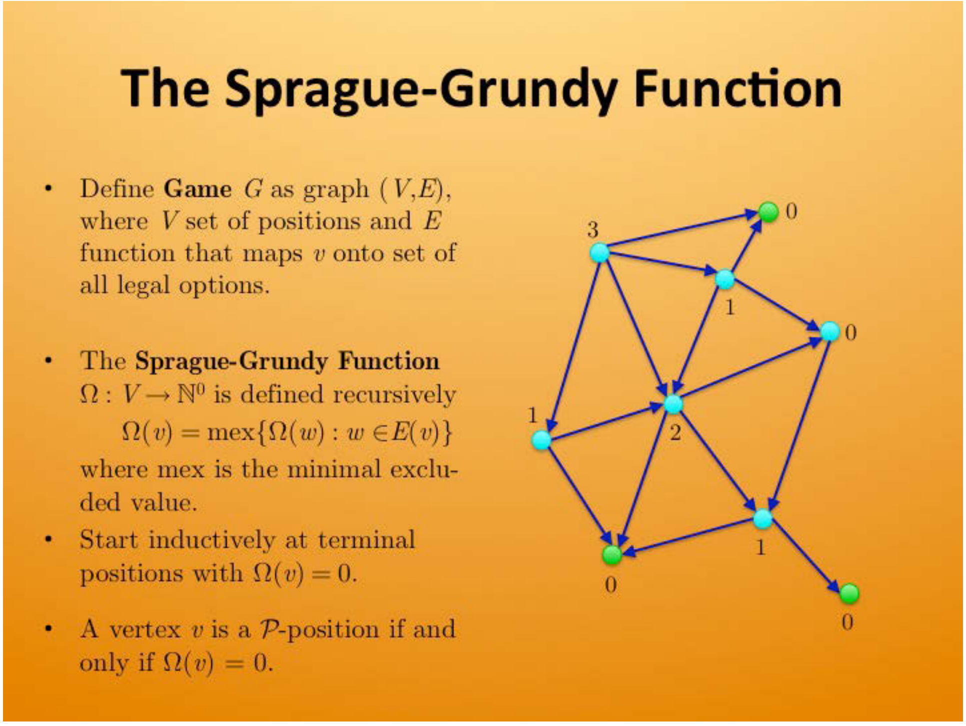


Sums of Games

We see that one-pile Nim is trivial and two-pile Nim is quite easy. But what about three or more piles? First, let us define what we mean by adding impartial games: The sum of two games G and H is just the union of their options sets. This means that you can imagine both games being played simultaneously and each time the player can decide whether to move in G or H. If a player can't move in both G and H, he has lost.

Therefore we can express a game of Nim with many piles as the sum of many single-pile games. For example, a (3,4,2) Nim game is equivalent to the sum of a (3), a (4) and a (2) Nim game. Those one-heap Nim games have a special name, Nimbers, and are denoted by $*n$.

We can easily determine whether a nimber is a $\mathcal{P}$ position or a $\mathcal{N}$ position. We now want to use this information to determine whether the sum of several nimbers is a $\mathcal{P}$ or a $\mathcal{N}$-position. Clearly, the sum of two $\mathcal{P}$-positions is a $\mathcal{P}$-position since to win you just apply you winning move in either game, depending on which game you opponent chooses. Similarly, the sum of a $\mathcal{P}$ and a $\mathcal{N}$-position is a $\mathcal{N}$-position, since the next player can move to a $\mathcal{P}$-$\mathcal{P}$ position and then proceed as above. However we can't yet determine the sum of two $\mathcal{N}$-positions.

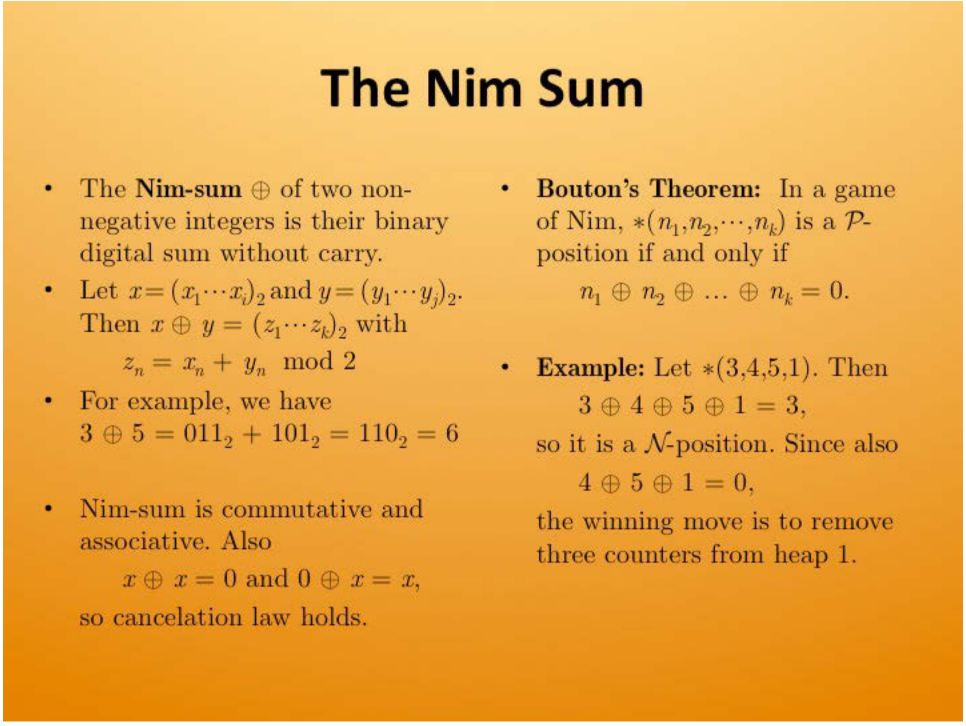


The Sprague-Grundy Functions

Let us find a new definition for impartial games: A Game G can be represented by a graph (V,E), where V is the set of all positions of the game (vertices of the graph) and E is a function that maps a position v to all legal options w (arrows/edges of the graph). Every impartial game can be expressed in such a way.

We now define the Sprague-Grundy Function, which assigns to each vertex of the graph a non-negative integer. We can find the Grundy value of any vertex v of the graph inductively: Consider this simple graph: All terminal positions have a Grundy value of 0. For all following positions, the Grundy value is the smallest non-negative integer not among the Grundy values of all options of the vertex. Formally, it is the minimal excluded value (mex) of all Grundy values of the options w of a vertex v.

From the definition it follows that from a position with Grundy value 0 you can only move to positions with a non-zero Grundy value and from any such position, you can move to at least one position with Grundy value 0. Therefore the definition of positions with Grundy value 0 is equivalent to the definition of $\mathcal{P}$-positions. However the Sprague-Grundy function includes much more information then only $\mathcal{P}$ and $\mathcal{N}$-positions.



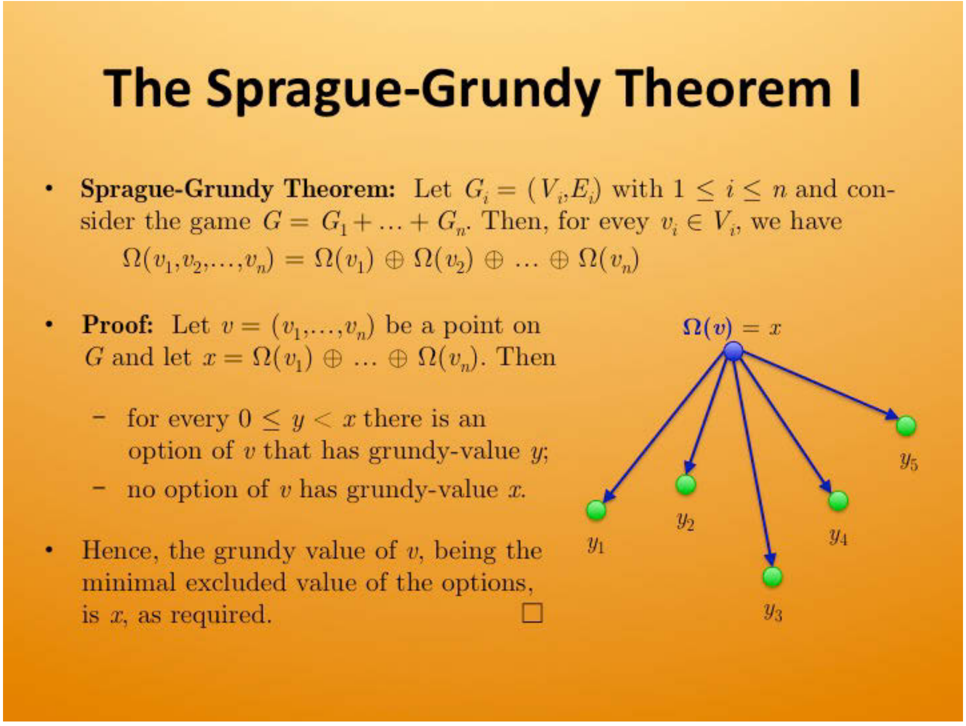
The Nim-Sum

To be able to add two games we also need a special operation, the Nim-sum denoted by $\oplus$. The Nim-sum of two non-negative integers is their binary digital sum without carry. For example, $3 \oplus 5 = 011 \oplus 101$ in binary $= 110$ in binary $= 6$. Formally if we write two non-negative integers x and y in binary, their Nim-sum is another non-negative integer z, with the nth binary digit being the sum of the nth binary digits of x and y mod 2.

Since addition mod 2 is commutative and associative, so is the Nim-sum. It is also follows from the definition that $x \oplus x = 0$, since $0 + 0 = 0$ and $1 + 1 = 0$ mod 2 and that $0 \oplus x = x$. Therefore the Nim-sum satisfies the cancellation laws.

Using the Nim-sum, we can now solve any game of Nim, because Bouton's Theorem (C. L. Bouton, 1902) states that any game of Nim with k piles of size $n_1, n_2, \ldots, n_k$ is a $\mathcal{P}$-position if and only is the Nim-sum of the individual pile sizes is 0.

For example, suppose we have a game (3,4,5,1). We can calculate that $3 \oplus 4 \oplus 5 \oplus 1 = 3 \neq 0$, therefore this is a $\mathcal{N}$-position. However, if it is our turn, we must be able to move to a P-position. We can do this easily by cancelling the 3 on both sides of the equation: Then $4 \oplus 5 \oplus 1 = 0$, so the winning move is to remove all three counters from heap 1.



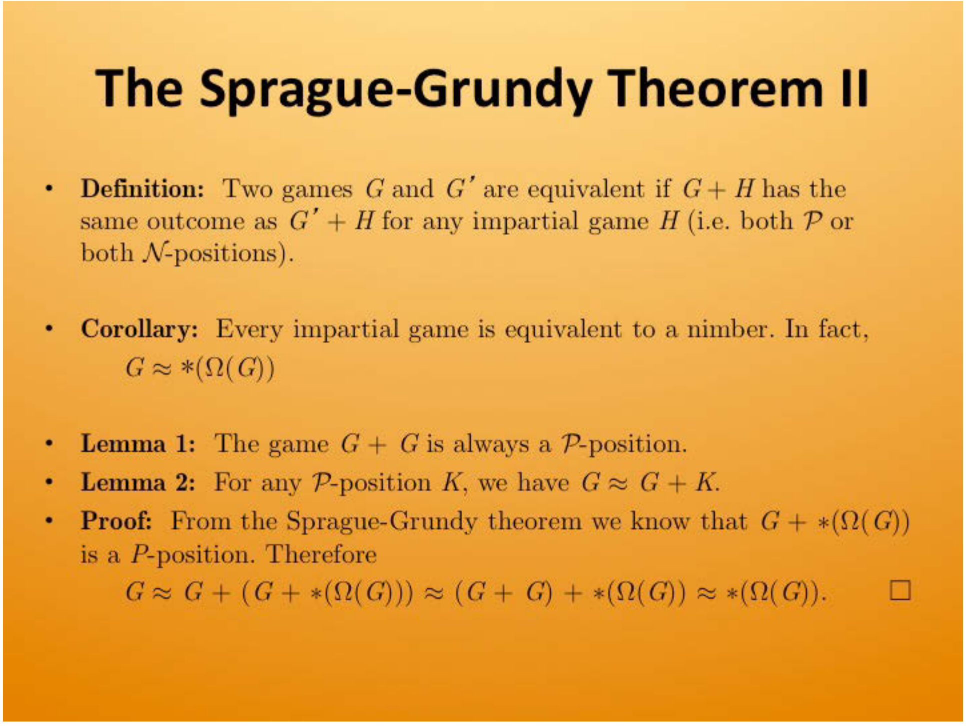
The Sprague-Gundy Theorem I

Instead of proving Bouton's theorem, I am going to outline the proof a much stronger theorem: The Sprague-Grundy theorem (R. P. Sprague, 1935; P. M. Grundy, 1939). Suppose we have n games G_i with graphs (V_i,E_i). Let v_i be any position of G_i. Then the Grundy value of the vertex $(v_1,\ldots v_n)$ in G is the Nim-sum of the Grundy values of the individual vertices v_i. Using this theorem, we can decompose any complicated impartial game into the sum of many easy games. We can find the Grundy-values for the vertices in these easy games and – using the Nim sum – we can hence calculate the Grundy-values of the complex game. Once we know the Grundy-values of a game it is not hard to find a winning strategy for the game.

Suppose that $v = (v_1,\ldots,v_n)$ is any vertex in the combined game G and let x be the Nim-sum of the Grundy-values of the individual vertices. Using induction, it is not hard to show the following two facts:
- For every non-negative integer y less than x there exists an option of v with Grundy value y.
- Furthermore, no option of v has Grundy value x.
Then the Grundy value of v is the minimal excluded value of the options, is x, as required.

Remark:
The full proof of both Bouton's Theorem and the Sprague Grundy Theorem can be found here:

Game Theory, T. S. Ferguson (UCLA)
[*http://www.math.ucla.edu/~tom/Game_Theory/comb.pdf*](http://www.math.ucla.edu/~tom/Game_Theory/comb.pdf)



The Sprague-Gundy Theorem II

To proceed, let us first define what we mean by two games to be equivalent: Two games G and G' are equivalent if $G + H$ and $G' + H$ have the same outcome for any impartial game H, i.e. if $G + H$ and $G' + H$ are either both $\mathcal{P}$-positions or both $\mathcal{N}$-positions.

And a corollary of the Sprague-Grundy theorem now explains why it is worth spending so much time thinking about the game of Nim. Every impartial game G is equivalent the nimber of size of the grundy value of G. Note that in this context, the game doesn't refer to a set of rules or positions, i.e. a graph, but to a certain position, i.e. the set of all options which are in turn set of options.

To proof this, we need two lemmas, which we actually have considered before: Firstly, $G + G$ is always a $\mathcal{P}$-positions, since the winning strategy for the previous player is just to repeat his opponents moves on the other game respectively. Then he will always make the last move. Secondly, if K is a $\mathcal{P}$-position, $G \approx G + K$, i.e. you can add as many $\mathcal{P}$-positions to a game as you want, without changing its outcome. This is true, since the sum of two $\mathcal{P}$-positions is a $\mathcal{P}$-position, and the sum of a $\mathcal{N}$ and a $\mathcal{P}$ position is a $\mathcal{N}$ position.

From the Sprague-Grundy Theorem we also know that $G +$ the nimber of size $\Omega(G)$ is a $\mathcal{P}$-position, since for any G, $\Omega(G) \oplus \Omega(G) = 0$. And now it is not hard to prove the corollary:

$$G \approx G + (G + *(\Omega(G))) \approx (G + G) + *(\Omega(G)) \approx *(\Omega(G)).$$

Therefore, just by knowing how to play Nim, we can find a winning strategy for any impartial game.

Grundy's Game

Grundy's Game is a variation of Nim. Again you start with some heaps of counters. Two players alternately split the heaps into two heaps of unequal size. The first player who is unable to move, i.e. when there are only heaps of sizes 1 and 2 left, loses.

We can easily decompose one such game consisting of a number of heaps as the sum of several single-pile versions of Grundy's game.

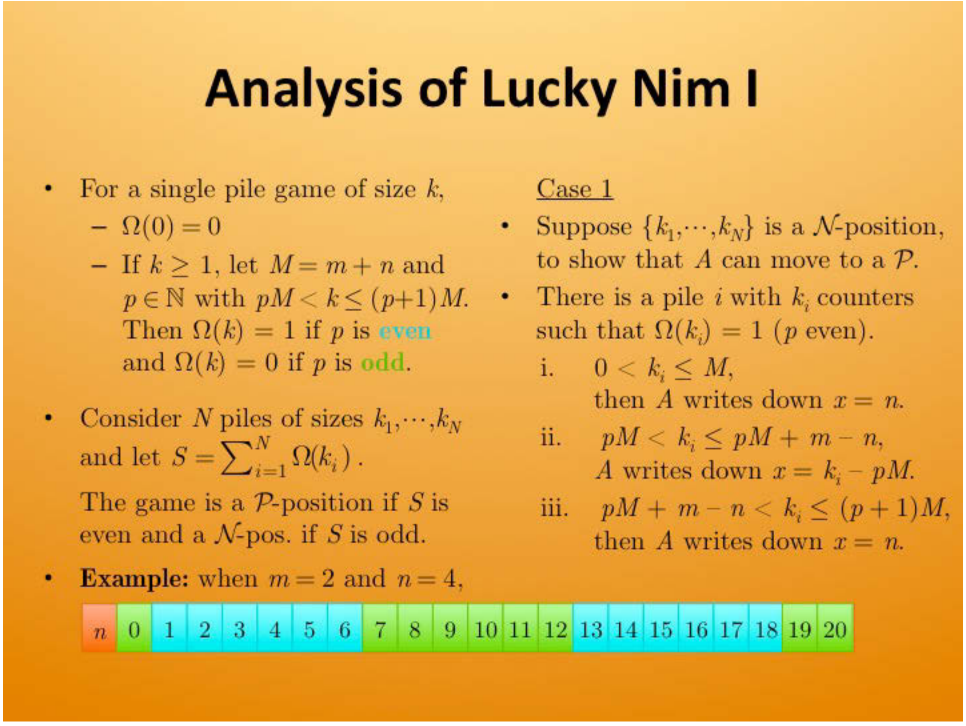
However, even with only one pile, Grundy's game proves to be quite complex: The first few Grundy values for one heap of size n are 0,0,1,0,2,1,0,2,1,0,… It is interesting to see how certain pattern repeat. This sequence has been calculated step by step up to $n = 2^{35}$ but is still not completely understood. The numbers increase very slowly and it is possible that they might become periodic after some time.

Lucky Nim

We have just spent a lot of time thinking about impartial games, that do not involve chance. As a contrast, consider the following game: Two players A and B independently lay down two coins. If both coins show the same side, A wins and if they show different sizes, B wins. It is obvious that the winning probability for either player is 50%. There is no mathematical strategy, you just have to be lucky.

Lucky Nim works similar to the normal game of Nim. Again we have a number of heaps with some counters. Suppose we also have two positive integers m and n with $m < n$. Two players A and B move alternately and their moves consist of the following steps: First, A chooses a non-empty pile, say of size k. Then, A and B both secretly chose two integers x and y respectively in the interval $[m,n]$ and show them simultaneously. This is the step corresponding to the tossing of coins. If now $x + y$ is less than k, A removes $x + y$ counters from the heap he chose, leaving $k - x - y$ counters. If $x + y \geq k$, A removes all counters from this heap making it empty. The move of B is the same, only that A and B and x and y are swapped around. The player to remove the last counter wins.

Since in the game of Nim there is usually onle one move which leads from a $\mathcal{N}$-position to a $\mathcal{P}$-position, and neither of the players knows each others number, this game seems to depend on pure chance – similar to tossing coins. However I will now show you how to find a winning strategy for the game.



Analysis of Lucky Nim I

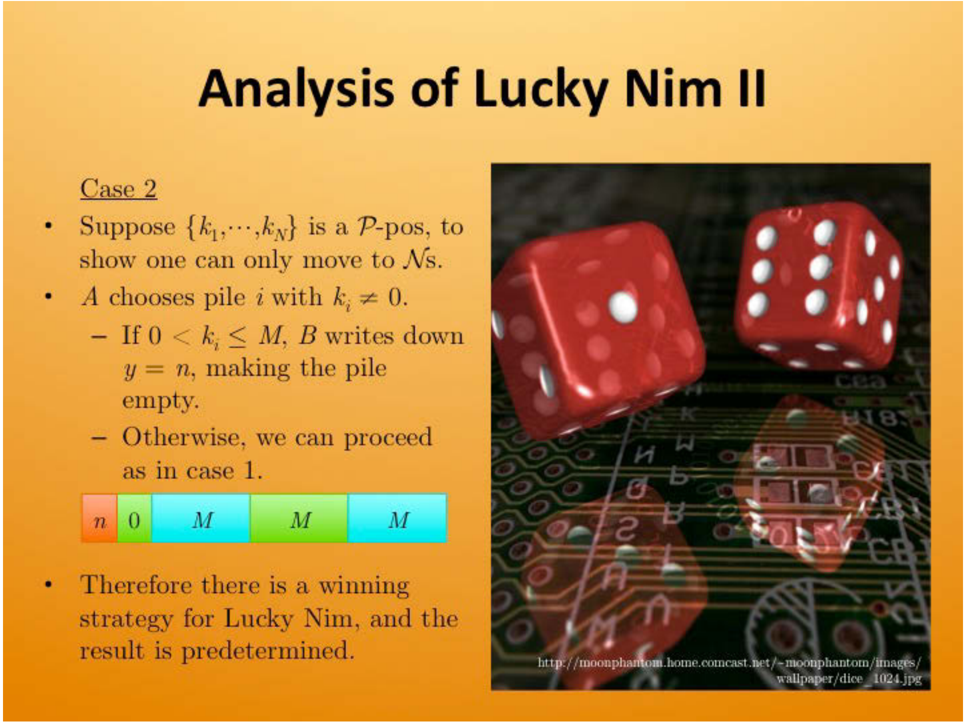
Treating the game as an impartial game, we first need to define which positions are $\mathcal{P}$ or N-positions, and then have to show that from any P-position you can only move to N-positions, and from any N-position you can move to at least one P-position.

First of all, consider a single pile game of size k. As usually, $\Omega(0) = 0$. For $k \geq 1$, let $M = m + n$ be the sum of the end-points of the interval in which A and B can choose numbers. We find a natural number p such that $pM < k \leq (p+1)M$. Then $\Omega(k) = 1$ if p is even and $\Omega(k) = 0$ if p is odd. This looks very complicated but is actually very simple: For example, if $m = 2$ and $n = 4$, i.e. $M = 6$, we have $\Omega(0) = 0$ by definition. Then we colour in blocks of 6 –numbers alternately, i.e. $\Omega(1)$ up to $\Omega(6) = 1$, $\Omega(7)$ up to $\Omega(12) = 0$ and so on.

Now suppose we have a game with $\mathcal{N}$ piles of sizes k_1 up to k_N. Then the overall game is a $\mathcal{P}$-position if the S, sum of all Ωs of the individual piles, is even and the game is a $\mathcal{N}$-position if S is odd.

We now have to consider two cases: starting on a $\mathcal{N}$-position and starting on a $\mathcal{P}$-position. Suppose first that we start on a $\mathcal{N}$-position and let A be the next player. Then we want to show that A can always move to a $\mathcal{P}$-position, no matter what B does. Since we are on a $\mathcal{N}$-position, to sum S of all Ω-values of all piles is odd, so there must be at least one pile i with k_i counters such that $\Omega(k_i) = 1$. Then the p above is even, so we are in one of the blue areas. In order to move to a $\mathcal{P}$-position, A must try to end his move in one of the green areas. In that case, the Ω value changes to 0, so the sum S becomes even.

We can see why this is always possible: Should we start in the first blue interval, A just writes down n. Since B has to choose a number $\geq m$, we will always end on 0 (or overshoot). If we were to start in the second interval, say on 17, A again writes down 4 and because B chooses a number ≥ 2, we will again move into the green interval. A only has to be careful not to overshoot: If we were to start on 14, for example, and A would write down 4, B could write down 4 as well and hence end up on 6 – in another blue interval. However it is not hard to write down the required inequalities and it is clear that it will always work.



Analysis of Lucky Nim II

Now suppose that we start on a $\mathcal{P}$ position k_1 up to k_N. Again suppose that it is As move. We now want to show that A will always move to a $\mathcal{N}$ position. Let A choose a non-empty pile i of size $k_i \neq 0$. This time we don't know whether the pile is "green" or "blue" but we want to show that B can always force the colour to change, i.e. the sum S to change by 1.

If $k_i < m + n$, B can write down $y = n$, hence making the pile empty and so moving to a $\mathcal{N}$-position. Otherwise, i.e. if $k_i > m + n$ we can proceed just as in case 1 to show the same results.

Therefore this chance game, which really didn't look like an impartial game at first sight, can be analysed using the same theory we have used for Nim and is completely determined.

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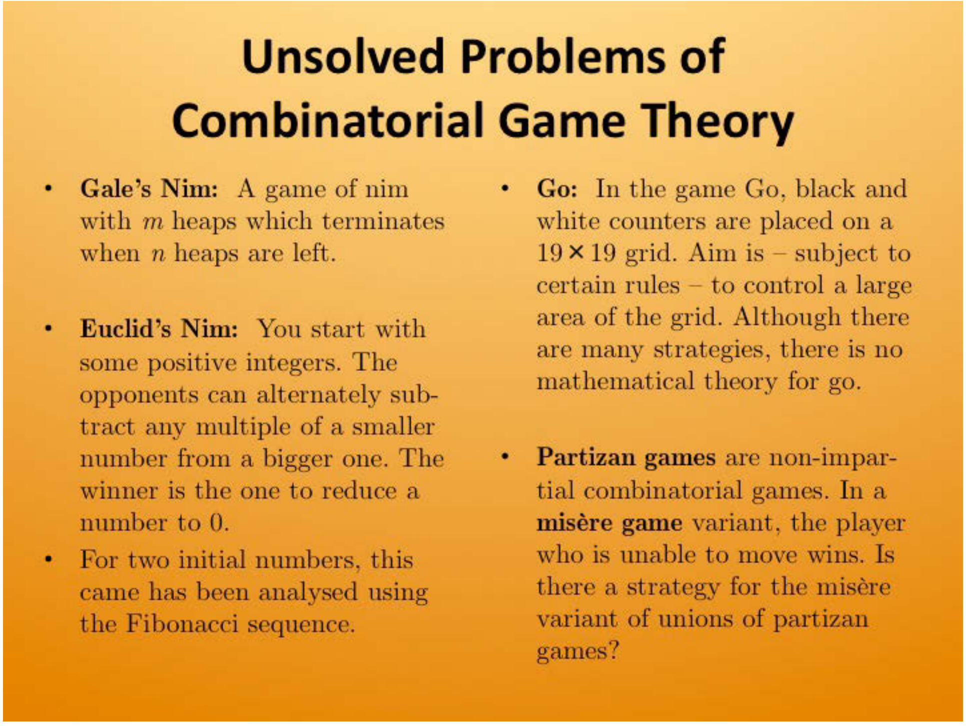

Unsolved Problems of Combinatorial Game Theory

There are many unsolved problems in Game theory. Indeed, you can just make up new games and – if you are lucky – the analysis is very interesting and might involve many different areas of mathematics, such as the Fibonacci sequence, Group theory, Prime numbers or Analysis. I have already mentioned that the grundy values of Grundy's game with one pile are not completely understood yet. Here are some other examples of unsolved problems in Combinatorial Game Theory:

Gale's Nim, for example, is a normal game of Nim, however you don't wait until all counters have disappeared but until only n heaps are left.

In Euclid's Nim, you start with some positive integers. The opponents alternately subtract ay multiple of a smaller of those numbers from a bigger one. The winner is the one to reduce the first number to 0. This game has been analysed for two initial numbers and the result involved Fibonacci numbers. It isn't known whether there is a strategy for more than two initial numbers.

Go is a very famous board-game, where black and white counters are placed on a 19*19 square grid. Both opponents try to control a larger area than each other. There are many interesting strategies and tricks, but no complete mathematical theory.

And one last, more theoretical unsolved problem: Find a strategy for the misère variant of unions of partizan games: Partizan games are non-impartial combinatorial games and the misère variant is when the player unable to move wins.

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